

Generalized Trade Reduction Mechanisms

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ABSTRACT

When designing a mechanism there are several desirable properties to maintain such as incentive compatibility (IC), individual rationality (IR), and budget balance (BB). It is well known [15] that it is impossible for a mechanism to maximize social welfare whilst also being IR, IC, and BB. There have been several attempts to circumvent [15] by trading welfare for BB, e.g., in domains such as double-sided auctions[13], distributed markets[3] and supply chain problems[2, 4].

In this paper we provide a procedure called a *Generalized Trade Reduction (GTR)* for *single-value players*, which given an IR and IC mechanism, outputs a mechanism which is IR, IC and BB with a loss of welfare. We bound the welfare achieved by our procedure for a wide range of domains. In particular, our results improve on existing solutions for problems such as double sided markets with homogenous goods, distributed markets and several kinds of supply chains. Furthermore, our solution provides budget balanced mechanisms for several open problems such as combinatorial double-sided auctions and distributed markets with strategic transportation edges.

Categories and Subject Descriptors

J.4 [Social and Behavioral Sciences]: Economics; K.4.4 [Electronic Commerce]: Payment scheme

General Terms

Algorithms, Design, Economics, Theory

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Trade Reduction, Budget Balance, Internal Competition, External Competition

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1. INTRODUCTION

When designing a mechanism there are several key properties that are desirable to maintain. Some of the more important ones are *individual rationality (IR)* - to make it worthwhile for all players to participate, *incentive compatibility (IC)* - to give incentive to players to report their true value to the mechanism and *budget balance (BB)* - not to run the mechanism on a loss. In many of the mechanisms the goal function that a mechanism designer attempts to maximize is the *social welfare*¹ - the total benefit to society. However, it is well known from [15] that any mechanism that maximizes social welfare while maintaining individual rationality and incentive compatibility runs a deficit perforce, i.e., is *not* budget balanced.

Of course, for many applications of practical importance we lack the will and the capability to allow the mechanism to run a deficit and hence one must balance the payments made by the mechanism. To maintain the BB property in an IR and IC mechanism it is necessary to compromise on the optimality of the social welfare.

1.1 Related Work and Specific Solutions

There have been several attempts to design budget balanced mechanisms for particular domains². For instance, for double-sided auctions where both the buyers and sellers are strategic and the goods are homogeneous [13] (or when the goods are heterogeneous [5]). [13] developed a mechanism that given valuations of buyers and sellers produces an *allocation* (which are the trading players) and a *matching* between buyers and sellers such that the mechanism is IR, IC, and BB while retaining most of the social welfare. In the distributed markets problem (and closely related problems) goods are transported between geographic locations while incurring some constant cost for transportation. [16, 9, 3] present mechanisms that approximate the social welfare while achieving an IR, IC and BB mechanism. For supply chain problems [2, 4] bounds the loss of social welfare that is necessary to inflict on the mechanism in order to achieve the desired combination of IR, IC, and BB.

Despite the works discussed above, the question of how to design a general mechanism that achieves IR, IC, and BB independently of the problem domain remains open. Furthermore, there are several domains where the question of how to design an IR, IC and BB mechanism which approx-

¹Social Welfare is also referred to as efficiency in the economics literature.

²A brief reminder of all of the problems used in this paper can be found in Appendix B

imates the social welfare remains an open problem. For example, in the important domain of combinatorial double-sided auctions there is no known result that bounds the loss of social welfare needed to achieve budget balance. Another interesting example is the open question left by [3]: How can one bound the loss in social welfare that is needed to achieve budget balance in an IR and IC distributed market where the transportation edges are strategic. Naturally an answer to the BB distributed market with strategic edges has vast practical implications, for example to transportation networks.

1.2 Our Contribution

In this paper we unify all the problems discussed above (both the solved as well as the open ones) into one solution concept procedure. The solution procedure called the *Generalized Trade Reduction (GTR)*. GTR accepts an IR and IC mechanism for *single-valued players* and outputs an IR, IC and BB mechanism. The output mechanism may suffer some welfare loss as a tradeoff of achieving BB. There are problem instances in which no welfare loss is necessary but by [15] there are problem instances in which there is welfare loss. Nevertheless for a wide class of problems we are able to bound the loss in welfare. A particularly interesting case is one in which the input mechanism is an efficient allocation.

In addition to unifying many of the BB problems under a single solution concept, the GTR procedure improves on existing results and solves several open problems in the literature. The existing solutions our GTR procedure improves are homogeneous double-sided auctions, distributed markets [3], and supply chain [2, 4]. For the homogeneous double-sided auctions the GTR solution procedure improves on the well known solution [13] by allowing for some cases of no trade reduction at all. For the distributed markets [3] and the supply chain [2, 4] the GTR solution procedure improves on the welfare losses' bound, i.e., allows one to achieve an IR, IC and BB mechanism with smaller loss on the social welfare. Recently we also learned that the GTR procedure allows one to turn the model newly presented [6] into a BB mechanism. The open problems that are answered by GTR are distributed markets with strategic transportation edges and bounded paths, combinatorial double-sided auctions with bounded size of the trading group i.e., a buyer and its bundle goods' sellers, combinatorial double-sided auctions with bounded number of possible trading groups.

In addition to the main contribution described above, this paper also defines an important classification of problem domains. We define *class based domain* and *procurement class based domains*. The above definitions build on the different competition "powers" of players in a mechanisms called *internal* and *external competition*. Most of the studied problem domains are of the more restrictive procurement class domains and we believe that the more general setting will inspire more research.

2. PRELIMINARIES

2.1 The Model

In this paper we design a method which given any IR and IC mechanism outputs a mechanism that maintains the IC and IR properties while achieving BB. For some classes of mechanisms we bound the competitive approximation of welfare.

In our model there are N players divided into sets of trade. The sets of trade are called *procurement sets* and are defined (following [2]) as follows:

DEFINITION 2.1. *A procurement set \mathbf{s} is the smallest set of players that is required for trade to occur.*

For example, in a double-sided auction, a procurement set is a pair consisting of a buyer and a seller. In a combinatorial double-sided auction a procurement set can consist of a buyer and several sellers. We mark the set of all procurement sets as \mathbf{S} and assume that any allocation is a disjoint union of procurement sets.

Each player i , $1 \leq i \leq n$, assigns a real value $v_i(\mathbf{s})$ to each possible procurement set $\mathbf{s} \in \mathbf{S}$. Namely, $v_i(\mathbf{s})$ is the valuation of player i on procurement set \mathbf{s} . We assume that for each player i $v_i(\mathbf{s})$ is i 's private value and that i is a *single value player*, meaning that if $v_i(\mathbf{s}_j) > 0$ then for every other \mathbf{s}_k , $k \neq j$, either $v_i(\mathbf{s}_k) = v_i(\mathbf{s}_j)$ or $v_i(\mathbf{s}_k) = 0$. For the ease of notation we will mark by v_i the value of player i for any procurement set \mathbf{s} such that $v_i(\mathbf{s}) > 0$. The set $V_i \subseteq \mathbb{R}$ is the set of all possible valuations v_i . The set of all possible valuations of all the players is denoted by $V = V_1 \times \dots \times V_n$. Let $v_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ be the vector of valuations of all the players besides player i , and let V_{-i} be the set of all possible vectors v_{-i} .

We denote by $W(\mathbf{s})$ the *value of a procurement set $\mathbf{s} \in \mathbf{S}$* such that $W(\mathbf{s}) = \sum_{i \in \mathbf{s}} v_i(\mathbf{s}) + F(\mathbf{s})$, where F is some function that assigns a constant to procurement sets. For example, F can be a (non-strategic) transportation cost in a distributed market problem. Let the *size of a procurement set \mathbf{s}* be denoted as $|\mathbf{s}|$.

It is assumed that any allocation is a disjoint union of procurement sets and therefore one can say that an allocation partitions the players into two sets; a set of players that trade and a set of players that do not trade.

The paper denotes by O the set of possible partitions of an allocation A into procurement sets. The *value $W(A)$ of an allocation A* is the sum of the values of its most efficient partition to procurement sets, that is $W(A) = \max_{S \in O} \sum_{\mathbf{s} \in S} W(\mathbf{s})$. This means that $W(A) = \sum_{i \in A} v_i + \max_{S \in O} \sum_{\mathbf{s} \in S} F(\mathbf{s})$. In the case where F is identically zero, then $W(A) = \sum_{i \in A} v_i$.

An *optimal partition $S^*(A)$* is a partition that maximizes the above sum for an allocation A . Let the value of A be $W(S^*(A))$ (note that the value can depend on F). We say that the allocation A is *efficient* if there is no other allocation with a higher value. The *efficiency* of the allocation \hat{A} is $\frac{W(\hat{A})}{W(A)}$, where A is a maximal valued allocation. We assume w.l.o.g. that there are no two allocations with the same value³.

A *mechanism M* defines an allocation and payment rules, $M = (R, P)$. A *payment rule P* decides i 's payment p_i where P is a function $P : V \rightarrow \mathbb{R}^N$. We work with mechanisms

³Ties can be broken using the identities of the players.

in which players are required to report their values. An example of such a mechanism is the VCG mechanism [17, 8, 10]. The reported value $b_i \in V_i$ of player i is called a *bid* and might be different from his private value v_i . Let $b \in V$ be the bids of all players. An *allocation rule* R decides the allocation according to the reported values $b \in V$. We make the standard assumption that players have *quasi-linear utility* so that when player i trades and pays p_i then his utility is $u_i(v_i, b_{-i}) = v_i - p_i$, $u_i : V \Rightarrow \mathbb{R}$. We also assume that players are *rational utility maximizers*.

Mechanism M is *Budget Balanced (BB)* if $\sum_{i \in N} p_i \geq 0$ for any bids $b \in V$. M is *Incentive-Compatible (IC) in dominant strategies* if for any player i , value v_i and any $b_{-i} \in V_{-i}$, $u_i(v_i, b_{-i}) \geq u_i(b)$ meaning that for any player i , bidding v_i maximized i 's utility over all possible bids of the other players. M is (ex-post) *Individually Rational (IR)* if for any player i value v_i , and any $b_{-i} \in V_{-i}$ $u_i(v_i, b_{-i}) \geq 0$ meaning that for all possible bids of the other players, player's i utility is non-negative. Note that since our mechanisms are normalized IR, if a player does not trade then the player pays 0 and has utility 0.

Our algorithm presented in the next section employs a commonly used payment scheme, the critical value payment scheme.

DEFINITION 2.2. *Critical value payment scheme: A mechanism uses a critical value payment scheme if given an allocation it charges players the minimum value they need to report to the mechanism in order to remain allocated.*

We denote by C_i the critical value price computed for player i .

2.2 Competitions and Domains

In this paper we present two generalized trade reduction algorithms. The two algorithms are such that given an IR and IC mechanism M that solves a problem in some domain (different domains are formally defined below), turns M into IR, IC and BB mechanism. The algorithm presented finds procurement sets and removes them in iterations until the ‘‘right conditions’’ are fulfilled and the mechanism M is turned into a BB one. The ‘‘right conditions’’ that need to be met are conditions of competition among the players in the given problem. The following definitions leads us to the competition conditions we are looking for.

DEFINITION 2.3. *For any player $i \in N$, we say that the set $R_i \subseteq N \setminus \{i\}$ is a **replacing set** of i , if for any procurement set $\mathbf{s} \in \mathbf{S}$ such that $i \in \mathbf{s}$ and $R_i \cap \mathbf{s} = \emptyset$, $\mathbf{s} \setminus \{i\} \cup R_i \in \mathbf{S}$.*

For example, in a (homogeneous) double-sided auction (see problem B.1) the replacement set for any buyer is simply any other buyer. In an auction for transportation slots (see problem B.7), the replacement set of an edge is a path between the endpoints of the edge. Note that a set can replace a single player. Furthermore, this relationship is transitive but *not* necessarily symmetric. If i is a replacement set for j , it is not necessarily true that j is a replacement set for i .

DEFINITION 2.4. *For any allocation A , procurement set $\mathbf{s} \subseteq A$, and any $i \in \mathbf{s}$ we say $R_i(A, \mathbf{s})$ is an **internal competition** for i with respect to A and \mathbf{s} , if $R_i(A, \mathbf{s}) \subseteq N \setminus A$ is a replacement set for i s.t. $T = \mathbf{s} \setminus \{i\} \cup R_i(A, \mathbf{s}) \in \mathbf{S}$ and $W(T) \geq 0$.*

DEFINITION 2.5. *For any allocation A and procurement set $\mathbf{s} \subseteq A$ and any $i \in \mathbf{s}$ we say that $E_i(A, \mathbf{s})$ is an **external competition** for i with respect to A and \mathbf{s} , if $E_i(A, \mathbf{s}) \subseteq N \setminus A$ is a set s.t., $T = \{i\} \cup E_i(A, \mathbf{s}) \in \mathbf{S}$ and $W(T) \geq 0$.*

We will assume, without loss of generality, that there are no ties between the values of any allocations, and in particular there are no ties between values of procurement sets. In case of ties, these can be broken by using the identities of the players⁴. So for any allocation A , procurement set \mathbf{s} and player i with external competition $E_i(A, \mathbf{s})$, there exists exactly one set representing the maximally valued external competition.

DEFINITION 2.6. *A set $X \subset N$ is closed under replacement if $\forall i \in X$ then $R_i \subset X$*

The following defines the required competition needed to maintain IC, IR and BB. The set X ⁵ denotes this competition and is closed under replacement. In the remainder of the paper we will assume that all of our sets which define competition in a mechanism are closed under replacement.

DEFINITION 2.7. *Let $X \subset N$ be a set that is closed under replacement, we say that the mechanism is an **X-external mechanism**, if*

1. Each player $i \in X$ has an external competition.
2. Each player $i \notin X$ has an internal competition.
3. For all players $i_1, \dots, i_t \in \mathbf{s} \setminus X$ there exist

$$R_{i_1}(A, \mathbf{s}), \dots, R_{i_t}(A, \mathbf{s})$$

such that for every $i_z \neq i_q$, $R_{i_z}(A, \mathbf{s}) \cap R_{i_q}(A, \mathbf{s}) = \emptyset$

4. for every procurement set $\mathbf{s} \in \mathbf{S}$ it holds that $\mathbf{s} \cap X \neq \emptyset$

For general domains the choice of X can be crucial. In fact even for the same domain the welfare (and revenue) can vary widely depending on how X is defined. In appendix C we give an example where two possible choices of X yield greatly different results. Although we show that X should be chosen as small as possible we do not give any characterization of the optimality of X and this is an important open problem.

Our two generalized trade reduction algorithms will ensure that for any allocation we have the desired types of competition. So given a mechanism M that is IC and IR with allocation A , the goal of the algorithms is to turn M into an X -external mechanism. The two generalized trade reduction algorithms utilize a *dividing function* D which divides allocation A into disjoint procurement sets. The algorithms order the procurements sets defined by D in order of increasing value. For any procurement set there is a desired type of competition that depends only on the players who compose the procurement set. The generalized trade reduction algorithms go over the procurement sets in order (from the smallest to the largest) and remove any procurement set that does not have the desired competition when the set is reached. The reduction of procurement sets will also be referred to as a trade reduction.

Formally,

⁴The details of how to break ties in allocations are standard and are omitted.

⁵We present some tradeoffs between the different possible sets in Appendix C.

DEFINITION 2.8. D is a **dividing function** if for any allocation A and the players' value vector v , D divides the allocation into disjoint procurements $\mathbf{s}_1, \dots, \mathbf{s}_k$ s.t. $\cup \mathbf{s}_j = A$ and for any player i with value v_i if $i \in \mathbf{s}_{j_1}$ and $t \in \mathbf{s}_{j_2}$ s.t. $j_1 \geq j_2$ then for any value $v'_i > v_i$ of player i and division by D into $\mathbf{s}'_1, \dots, \mathbf{s}'_{k'}$ such that $i \in \mathbf{s}'_{j'_1}$ and $t \in \mathbf{s}'_{j'_2}$ then $j'_1 > j'_2$.

The two generalized trade reduction algorithms presented accept problems in different domains. The formal domain definitions follow:

DEFINITION 2.9. A domain is a class domain if for all $i \in N$ and all replacement sets of i , R_i , $|R_i| = 1$ and for all i, j , $i \neq j$ if $j = R_i$ then $i = R_j$.

Intuitively, this means that replacement sets are of size 1 and the replacing relationship is symmetric.

We define the class of a player i as the set of the player's replacement sets and denote the class of player i by $[i]$. It is important to note that since replacement sets are transitive relations and since class domains also impose symmetric relations on the replacement sets, the class of a player i , $[i]$ is actually an equivalence class for i .

DEFINITION 2.10. A domain is a procurement-class domain if the domain is a class-based domain and if for any player i such that there exists two procurement sets $\mathbf{s}_1, \mathbf{s}_2$ (not necessarily trading simultaneously in any allocation) such that $i \in \mathbf{s}_1$ and $i \in \mathbf{s}_2$ then there exists a bijection $f: \mathbf{s}_1 \rightarrow \mathbf{s}_2$ such that for any $j \in \mathbf{s}_1$, $f(j)$ is a replacement set for j in \mathbf{s}_2 .

EXAMPLE 2.1. A (homogeneous) double-sided auction (see problem B.1) is a procurement-class based domain. For the (homogeneous) double-sided auction each procurement set consists of a buyer and a seller.

The double sided combinatorial auction consisting of a single multi-minded buyer and multiple sellers of heterogenous goods (see problem B.9), is a class based domain (as we have a single buyer) but not a procurement-class based domain. In this case, the buyer is a class and each set of sellers of the same good is a class. However, for a buyer there is no bijection between the different the procurement sets of the bundles of goods the buyer is interested in.

The spatial-distributed market with strategic edges (see problem B.6) is not a class-based domain (and therefore not a procurement-class domain). For example, even for a fixed buyer and a fixed seller there are two different procurement sets consisting of different paths between the buyers and sellers.

The next sections present two algorithms GTR-1 and GTR-2. GTR-1 accepts problems in procurement-class based domains, its properties are proved with a general dividing function D . The GTR-2 algorithm accepts problems in any domain. We prove the GTR-2's properties with specific dividing function D_0 . The function will be defined in section 4. Since the dividing function can have a large practical impact on welfare (and revenue) the generality of GTR – 1 (albeit in special domains) can be an important practical consideration.

3. PROCUREMENT-CLASS BASED DOMAINS

This section focuses on the problems that are procurement-class based domains. For this domain, we present an algorithm called GTR-1, which given a mechanism that is IR and IC outputs a mechanism with reduced welfare which is IR, IC and budget balanced.

Although procurement class domains appear to be a relatively restricted model, in fact many domains studied in the literature are procurement class domains.

EXAMPLE 3.1. The following domains are procurement class domains:

- double-sided auctions with homogenous goods [13](problem B.1). In this domain there are two classes. The class of buyers and the class of sellers. Each procurement set consists of a single buyer and a single seller. Since every pair of (buyer, seller) is a valid procurement set (albeit possible with negative value) this is a procurement class domain. In this domain the constant assigned to the procurement sets is $F = 0$.
- Spatially distributed markets with non strategic edges [3, 9](problem B.3). Like the double-sided auctions with homogenous goods, there are two classes in the domain. Class of buyers and class of sellers with procurement sets consisting of single buyer and single seller. The sellers and buyers are nodes in a graph and the function F is the distance of two nodes (length of the edge) which represent transport costs. These costs differ between different (buyer, seller) pairs.
- Supply chains [2, 4] (problem B.5). The assumption of unique manufactory by [2, 4] can best be understood as turning general supply chains (which need not be a procurement class domain) into a procurement class domain.
- Single minded combinatorial auctions [11] (problem B.8). In this context each seller sells a single good and each buyer wants a set of goods. The classes are the sets of sellers selling the same good as well as the buyers who desire the same bundle. A procurement set consists of a single buyer as well as a set of sellers who can satisfy that buyer.

A definition of the mechanism follows:

DEFINITION 3.1. **The GTR-1 algorithm** - given a mechanism M , a set $X \subset N$ which is closed under replacement, a dividing function D , and allocation A , GTR-1 operates as follows:

1. Use the dividing function D to divide A into procurement sets $\mathbf{s}_1, \dots, \mathbf{s}_k \in \mathbf{S}$.
2. Order the procurement sets by increasing value.
3. For each \mathbf{s}_j , starting from the lowest value procurement set:
If for every $i \in \mathbf{s}_j \cap X$ there is external competition and every $i \in \mathbf{s}_j \setminus X$ there is internal competition then

keep \mathbf{s}_j . Otherwise reduce the trade \mathbf{s}_j (i.e., remove every $i \in \mathbf{s}_j$ from the allocation).⁶

4. All trading players are charged the critical value for trading. All non trading players are charged nothing.

REMARK 3.1. *The special case where $X = N$ has received attention under different guises in various special cases, such as ([13, 3, 4]).*

3.1 The GTR-1 Produces an X -external Mechanism that is IR, IC and BB

In this subsection we prove that the GTR-1 algorithm produces an X -external mechanism that is IR, IC and BB.

To prove GTR-1's properties we make use of theorem 3.1 which is a well known result (e.g., [14, 11]). Theorem 3.1 characterizes necessary and sufficient conditions for a mechanism for single value players to be IR and IC:

DEFINITION 3.2. *An allocation rule R is Bid Monotonic if for any player i , any bids of the other players $b_{-i} \in V_{-i}$, and any two possible bids of i , $\hat{b}_i > b_i$, if i trades under the allocation rule R when reporting b_i , then i also trades when reporting \hat{b}_i .*

Intuitively, a bid monotonic allocation rule ensures that no trading player can become a non-trading player by improving his bid.

THEOREM 3.1. *An IR mechanism M with allocation rule R is IC if and only if R is Bid Monotonic and each trading player i pays his critical value C_i ($p_i = C_i$).*

So for normalized IR⁷ and IC mechanisms, the allocation rule which is bid monotonic uniquely defines the critical values for all the players and thus the payments.

OBSERVATION 3.1. *Let M_1 and M_2 be two IR and IC mechanisms with the same allocation rule. Then M_1 and M_2 must have the same payment rule.*

In the following we prove that the X -external GTR-1 algorithm produces a IR, IC and BB mechanism, but first a subsidiary lemma is shown.

LEMMA 3.1. *For procurement class domains if there exists a procurement set \mathbf{s}_j s.t. $i \in \mathbf{s}_j$ and i has external competition than all $t \neq i$ $t \in \mathbf{s}_j$, t has internal competition.*

PROOF. This follows from the definition of procurement class domains. Suppose that i has external competition, then there exists a set of players $E_i(A, \mathbf{s})$ such that $\{i\} \cup E_i(A, \mathbf{s}) \in \mathbf{S}$. Let us denote by $\mathbf{s}'_j = \{i\} \cup E_i(A, \mathbf{s})$. Since the domain is a procurement-class domain there exists a bijection function f between \mathbf{s}_j and \mathbf{s}'_j . f defines the required internal competition. \square

We start by proving IR and IC:

⁶Although the definition of an X -external mechanism requires that X intersects every procurement set, this is not strictly necessary. It is possible to define an X that does not intersect every possible procurement set. In this case, any procurement set $\mathbf{s} \in \mathbf{S}$ s.t. $\mathbf{s} \cap X = \emptyset$ will be reduced.

⁷Note that this is not true for mechanisms which are not normalized e.g., [7, 12]

LEMMA 3.2. *For any X , the X -external mechanism with a critical value pricing scheme produced by the GTR-1 algorithm is an IR and IC mechanism.*

PROOF. By the definition of a critical value pricing scheme 2.2 and the GTR-1 algorithm 3.1 it follows that for every trading player i , $v_i \geq 0$. By the GTR-1 algorithm 3.1 non-trading players i have a payment of zero. Thus for every player i , value v_i , and any $b_{-i} \in V_{-i}$ $u_i(v_i, b_{-i}) \geq 0$, meaning the produced X -external mechanism is IR.

As the X -external GTR-1 algorithm is IR and applies the critical value payment scheme according to theorem 3.1, in order to show that the produced X -external mechanism with the critical value payment scheme is IC, it remains to show that the produced mechanism's allocation rule is bid monotonic.

Since GTR-1 orders the procurement sets according to increasing value, if player i increases his bid from b_i to $b'_i > b_i$ then for any division function D of procurement sets, the procurement set \mathbf{s} containing i always appears later with the bid b'_i than with the bid b_i . So the likelihood of competition can only increase if i appears in later procurement sets. This follows as GTR-1 can reduce more of the lower value procurement sets which will result in more non-trading players.

Therefore if \mathbf{s} has the required competition and is not reduced with b_i then it will have the required competition with b'_i and will not be reduced. \square

Finally we prove BB:

LEMMA 3.3. *For any X , the X -external mechanism with critical value pricing scheme produced by the GTR-1 algorithm is a BB mechanism.*

PROOF. In order to show that the produced mechanism is BB we show that each procurement set that is not reduced has a positive budget (i.e., the sum of payments is positive).

Let $\mathbf{s} \in \mathbf{S}$ be a procurement set that is not reduced. Let $i \in \mathbf{s} \cap X$ then according to the definition of X -external definition 2.7 and the GTR-1 algorithm 3.1 i has an external competition. Assume w.l.o.g.⁸ that i is the only player with external competition in \mathbf{s} and all other players $j \neq i$, $j \in \mathbf{s}$ have internal competition.

Let A be the allocation after the procurement sets reduction by the GTR-1 algorithm. According to the definition of external competition 2.5, there exists a set $E_i(A, \mathbf{s}) \subset N \setminus A$ such that $i \cup E_i(A, \mathbf{s}) \in \mathbf{S}$ and $W(i \cup E_i(A, \mathbf{s})) \geq 0$. Since $W(i \cup E_i(A, \mathbf{s})) = v_i + W(E_i(A, \mathbf{s}))$ then $v_i \geq -W(E_i(A, \mathbf{s}))$. By the critical value pricing scheme definition 2.2 it means that if player i bids any less than $-W(E_i(A, \mathbf{s}))$ he will not have external competition and therefore will be removed from trading. Thus i pays no less than $\min -W(E_i(A, \mathbf{s}))$. Since all other players $j \in \mathbf{s}$ have internal competition their critical price can not be less than their maximal value internal competitor (set) i.e., $\max W(R_j(A, \mathbf{s}))$. If any player $j \in \mathbf{s}$ bids less then its maximal internal competitor (set) then he will not be in \mathbf{s} but his maximal internal competitor (set) will.

As a possible $E_i(A, \mathbf{s})$ is $\cup_{j \in \mathbf{s}} R_j(A, \mathbf{s})$ one can bound the maximal value of i 's external competition $W(E_i(A, \mathbf{s}))$ by the sum of the maximal values of the rest of the players in \mathbf{s}

⁸since the domain is a procurement class domain we can use lemma 3.1

internal competition i.e., $\sum_{j \in \mathbf{s}} \max W(R_j(A, \mathbf{s}))$. Therefore $\min -W(E_i(A, \mathbf{s})) = -(\sum_{j \in \mathbf{s}} \max W(R_j(A, \mathbf{s})))$. As the F function is defined to be a positive constant we get that $W(\mathbf{s}) = \min -W(E_i(A, \mathbf{s})) + (\sum_{j \in \mathbf{s}} \max W(R_j(A, \mathbf{s}))) + F(\mathbf{s}) \geq 0$ and thus \mathbf{s} is at least budget balanced. As each procurement set that is not reduced is at least budget balanced, it follows that the produced X -external mechanism is BB. \square

The above two lemmas yield the following theorem:

THEOREM 3.2. *For procurement class domains for any X , the X -external mechanism with critical value pricing scheme produced by the GTR-1 algorithm is an IR, IC and BB mechanism.*

REMARK 3.2. *The proof of the theorem yields bounds on the payments any player has to make to the mechanism.*

4. NON PROCUREMENT-CLASS BASED DOMAINS

The main reason that GTR-1 works for the procurement-class domains is that each player's possibility of being reduced is monotonic. By the definition of a dividing function if a player $i \in \mathbf{s}_j$ increases his value, i can only appear in later procurement set \mathbf{s}'_j and hence has a higher chance of having the desired competition.

Therefore, the chance of i lacking the requisite competition is decreased. Since the domain is a procurement class domain, all other players $t \neq i, t \in \mathbf{s}'_j$ are also more likely to have competition since members of their class continue to appear before i and hence the likelihood that i will be reduced is decreased. Since by theorem 3.1 a necessary and sufficient condition for the mechanism to be IC is monotonicity. GTR-1 is IC for procurement-class domains.

However, for domains that are *not* procurement class domains this does not suffice even if the domain is a class based domain. Although, all members of \mathbf{s}_j continue to have the required competition it is possible that there are members of \mathbf{s}'_j who do not have analogues in \mathbf{s}_j who do not have competition. Hence i might be reduced after increasing his value which by lemma 3.1 means the mechanism is not IC.

We therefore define a different algorithm for non procurement class domains.

Our modified algorithm requires a special dividing function in order to maintain the IC property. Although our restriction to this special dividing function appears stringent, the dividing function we use is a generalization of the way that procurement sets are chosen in procurement-class based domains e.g., [13, 16, 9, 3, 2, 4].

For ease of presentation in this section we assume that $F = 0$.

The dividing function for general domains is defined by looking at all possible dividing functions. For each dividing function D_i and each set of bids, the GTR-1 algorithm yields a welfare that is a function of the bids and the dividing function⁹. We denote by D_0 the dividing function that divides the players into sets s.t. the welfare that GTR-1 finds is maximal¹⁰.

⁹Note that for any particular D_i this might not be IC as GTR-1 is IC only for procurement class domains and not for general domains

¹⁰In Appendix A we show how to calculate D_0 in polyno-

Formally,

Let \mathcal{D} be the set of all dividing functions D . Denote the welfare achieved by the mechanism produced by GTR-1 when using dividing function D and a set of bids \bar{b} by $GTR1(D, \bar{b})$. Denote by $D_0(\bar{b}) = \operatorname{argmax}_{D \in \mathcal{D}} (GTR1(D, \bar{b}))$. For ease of presentation we denote $D_0(\bar{b})$ by D_0 when the dependence on b is clear from the context.

REMARK 4.1. *D_0 is an element of the set of dividing functions, and therefore is a dividing function.*

The second generalized trade reduction algorithm GTR-2 follows.

DEFINITION 4.1. The GTR-2 algorithm - *Given mechanism M , allocation A , and a set $X \subset N$ closed under replacement, GTR-2 operates as follows:*

1. Calculate the dividing function D_0 as defined above.
2. Use the dividing function D_0 to divide A into procurement sets $\mathbf{s}_1, \dots, \mathbf{s}_k \in \mathbf{S}$.
3. For each \mathbf{s}_j , starting from the lowest value procurement set, do the following:
If for $i \in \mathbf{s}_j \cap X$ there is an external competition and there is at most one $i \in \mathbf{s}_j$ that does not have an internal competition then keep \mathbf{s}_j . Otherwise, reduce the trade \mathbf{s}_j .
4. All trading players are charged the critical value for trading. All non trading players are charged zero.

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We will prove that the mechanism produced by GTR-2 maintains the desired properties of IR, IC, and BB. The following lemma shows that the GTR-2 produced mechanism is IR, and IC.

LEMMA 4.1. *For any X , the X -external mechanism with critical value pricing scheme produced by the GTR-2 algorithm is an IR and IC mechanism.*

PROOF. By theorem 3.1 it suffices to prove that the produced mechanism by the GTR-2 algorithm is bid monotonic for every player i . Suppose that i was not reduced when bidding b_i we need to prove that i will not be reduced when bidding $b'_i > b_i$. Denote by $D_1 = D_0(b)$ the dividing function used by GTR-2 when i reported b_i and the rest of the players reported b_{-i} . Denote by $D'_1 = D_0(b'_i, b_{-i})$ the dividing function used by GTR-2 when i reported b'_i and the rest of the players reported b_{-i} . Denote by $\bar{D}_1(b)$ a maximal dividing function that results in GTR-1 reducing i when reporting b_i . Assume to the contrary that GTR-2 reduced i from the trade when i reported b'_i then $GTR1(D'_1, (b'_i, b_{-i})) = GTR1(\bar{D}_1, b)$. Since $D_1 \in \mathcal{D}$ it follows that $GTR1(D_1, b) > GTR1(\bar{D}_1, b)$ and therefore $GTR1(D_1, b) > GTR1(D'_1, (b'_i, b_{-i}))$. However according to the definition $D'_1 \in \mathcal{D}$, GTR-2 should not have reduced i in polynomial time for procurement-class domains. Calculating D_0 in polynomial time for general domains is an important open problem.

¹¹In the full version GTR-2 is extended such that it suffices that there exists *some* time in which the third step holds. That extension is omitted from current version due to lack of space.

with the dividing function D'_1 and gained a greater welfare than $GTR1(D_1, b)$. Thus a contradiction arises and $GTR-2$ does not reduce i from the trade when i reports $b'_i > b_i$. \square

LEMMA 4.2. *For any X , the X -external mechanism with critical value pricing scheme produced by the $GTR-2$ algorithm is a BB mechanism.*

PROOF. This proof is similar to the proof of lemma 3.3. \square

Combining the two lemmas above we get:

THEOREM 4.1. *For any X closed under replacement, the X -external mechanism with critical value pricing scheme produced by the $GTR-2$ algorithm is an IR , IC and BB mechanism.*

Appendix A shows how to calculate D_0 for procurement class domains in polynomial time, it is not generally known how to easily calculate D_0 . Creating a general method for calculating the needed dividing function in polynomial time remains as an open question.

4.1 Bounding the Welfare for Procurement-Class Based Domains and other General Domains Cases

This section shows that in addition to producing a mechanism with the desired properties, $GTR-2$ also produces a mechanism that maintains high welfare. Since the $GTR-2$ algorithm finds a budget balanced mechanism in arbitrary domains we are unable to bound the welfare for general cases. However we can bound the welfare for procurement-class based domain and a wide variety of cases in general domains which includes many cases previously studied.

DEFINITION 4.2. *Denote $freq_k([i], \mathbf{s}_j)$ to indicate that a class $[i]$ appears in a procurement set \mathbf{s}_j , k times and there are k members of $[i]$ in \mathbf{s}_j .*

DEFINITION 4.3. *Denote by $freq_k([i], \mathbf{S})$ the maximal k s.t. there are k members of $[i]$ in \mathbf{s}_j . I.e., $freq_k([i], \mathbf{S}) = \max_{\mathbf{s}_j \in \mathbf{S}} freq_k([i], \mathbf{s}_j)$.*

Let the set of equivalence classes in procurement class based domain mechanism be e_c and $|e_c|$ be the number of those equivalence classes.

Using the definition of class appearance frequency we can bound the welfare achieved by the $GTR-2$ produced mechanism for procurement class domains¹²:

LEMMA 4.3. *For procurement class domains with $F = 0$, the number of procurement sets that are reduced by $GTR-2$ ¹³ is at most $|e_c|$ times the maximal frequency of each class. Formally, the maximal number of procurement sets that is reduced is $O(\sum_{[i] \in e_c} freq_k([i], \mathbf{S}))$*

PROOF. Let D be an arbitrary dividing function. We note that by definition any procurement set \mathbf{s}_j will not be reduced if every $i \in \mathbf{s}_j$ has both internal competition and external competition.

¹²The welfare achieved by $GTR-1$ can also be bounded for the cases presented in this section. However, we focus on $GTR-2$ as it always achieves better welfare.

¹³or $GTR-1$

Every procurement set \mathbf{s} that is reduced has at least one player i who has no competition. Once \mathbf{s} is reduced all players of $[i]$ have internal competition. So by reducing the number of equivalence classes $|e_c|$ procurement sets we cover all the remaining players with internal competition.

If the maximal frequency of every equivalence classes was one then each remaining player t in procurement set \mathbf{s}_k also have external competition as all the internal competitors of players $\bar{t} \neq t$, $\bar{t} \in \mathbf{s}_k$ are an external competition for t .

If we have $freq_k([t], \mathbf{S})$ players from class $[t]$ who were reduced then there is sufficient external competition for all players in \mathbf{s}_k .

Therefore it suffices to reduce $O(\sum_{[i] \in e_c} freq_k([i], \mathbf{S}))$ procurement sets in order to ensure that both the requisite internal and external competition exists. \square

The next theorem follows as an immediate corollary for lemma 4.3:

THEOREM 4.2. *Given procurement-class based domain mechanisms with H procurement sets, the efficiency is at least a $1 - O(\frac{O(\sum_{[i] \in e_c} freq_k([i], \mathbf{S}))}{H})$ fraction of the optimal welfare.*

The following corollaries are direct results of theorem 4.2. All of these corollaries either improve prior results or achieve the same welfare as prior results.

COROLLARY 4.1. *Using $GTR-2$ for homogenous double-sided auctions (problem B.1) at most¹⁴ one procurement set must be reduced.*

Similarly, for spatially distributed markets without strategic edges (problem B.3) using $GTR-2$ improves the result of [3] where a minimum cycle including a buyer and seller is reduced.

COROLLARY 4.2. *Using $GTR-2$ for spatially distributed markets without strategic edges at most one cycle per connected component¹⁵ will be reduced.*

For supply chains (problem B.5) using $GTR-2$ improves the result of [2, 4] similar to corollary 4.2.

COROLLARY 4.3. *Using $GTR-2$ for supply chains at most one cycle per connected component¹⁶ will be reduced.*

The following corollary solves the open problem at [3].

COROLLARY 4.4. *For distributed markets on n nodes with strategic agents and paths of bounded length K (problem B.6) it suffices to remove at most $K * n$ procurements sets.*

PROOF. Sketch: These will create at least K spanning trees, hence we can disjointly cover every remaining procurement set. This improves the naive algorithm of reducing n^2 procurement sets. \square

We provide results for two special cases of double sided CA with single value players (problem B.8).

¹⁴It is possible that no reductions will be made, for instance when there is a non-trading player who is the requisite external competition.

¹⁵Similar to the double-sided auctions, sometimes there will be enough competition without a reduction.

¹⁶Similar to double-sided auctions, sometimes there will be enough competition without a reduction.

COROLLARY 4.5. *if there are at most M different kinds of procurement sets it suffices to remove M procurement sets.*

COROLLARY 4.6. *If there are K types of goods and each procurement set consists of at most one of each type it suffices to remove at most K procurement sets.*

5. CONCLUSIONS AND FUTURE WORK

In this paper we presented a general solution procedure called the *Generalized Trade Reduction (GTR)*. GTR accepts an IR and IC mechanism as an input and outputs mechanisms that are IR, IC and BB. The output mechanisms achieves welfare that is close to optimal for a wide range of domains.

The GTR procedure improves on existing results such as homogeneous double-sided auctions, distributed markets, and supply chains, and solves several open problems such as distributed markets with strategic transportation edges and bounded paths, combinatorial double-sided auctions with bounded size procurements sets, and combinatorial double-sided auctions with a bounded number of procurement sets.

The question of the quality of welfare approximation both in general and in class domains that are not procurement class domains is an important and interesting open question. We also leave open the question of upper bounds for the quality of approximation of welfare. Although we know that it is impossible to have IR, IC and BB in an efficient mechanism it would be interesting to have an upper bound on the approximation to welfare achievable in an IR, IC and BB mechanism.

The GTR procedure outputs a mechanism which depends on a set $X \subset N$. Another interesting question is what the quality of approximation is when X is chosen randomly from N before valuations are declared.

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6. REFERENCES

- [1] A. Archer and E. Tardos. Frugal path mechanisms. *Symposium on Discrete Algorithms, Proceedings of the thirteenth annual ACM-SIAM symposium on Discrete algorithms*,2002.
- [2] M. Babaioff and N. Nisan. Concurrent Auctions Across the Supply Chain. *Journal of Artificial Intelligence Research*,2004.
- [3] Babaioff M., Nisan N. and Pavlov E. Mechanisms for a Spatially Distributed Market. *In proceedings of the 5th ACM Conference on Electronic Commerce*,2004.
- [4] M. Babaioff and W. E. Walsh. Incentive-Compatible, Budget-Balanced, yet Highly Efficient Auctions for Supply Chain Formation. *In proceedings of Fourth ACM Conference on Electronic Commerce*,2003.
- [5] Y. Bartal, R. Gonen and P. La Mura. Negotiation-range mechanisms: exploring the limits of truthful efficient markets. *EC '04: Proceedings of the 5th ACM conference on Electronic commerce*, 2004.
- [6] Blume L, Easley D., Kleinberg J. and Tardos E. Trading Networks with Price-Setting Agents. *In proceedings of the 8th ACM conference on Electronic commerce*,2007.

- [7] Cavallo R. Optimal decision-making with minimal waste: Strategyproof redistribution of VCG payments. *In Proc. 5th Int. Conf. on Auton. Agents and Multi-Agent Systems (AAMAS06)*.
- [8] E. H. Clarke Multipart Pricing of Public Goods. *In journal Public Choice 1971*, vol. 2, pp. 17-33.
- [9] Chu L. Y. and Shen Zuo-Jun M. Agent Competition Double Auction Mechanism. *Management Science*,vol 52(8),2006.
- [10] T. Groves Incentives in teams. *In journal Econometrica 1973*, vol. 41, pp. 617-631.
- [11] D. Lehmann, L. I. O'Callaghan, and Y. Shoham. Truth Revelation in Approximately Efficient Combinatorial Auctions. *In Journal of ACM 2002*, vol. 49(5), pp. 577-602.
- [12] Leonard H. Elicitation of Honest Preferences for the Assignment of Individuals to Positions. *Journal of political econ*,1983.
- [13] McAfee R. P. A Dominant Strategy Double Auction. *Journal of Economic Theory*,vol 56, 434-450, 1992.
- [14] A. Mu'alem, and N. Nisan. Truthful Approximation Mechanisms for Restricted Combinatorial Auctions. *Proceeding of AAAI 2002*.
- [15] Myerson R. B. and Satterthwaite M. A. Efficient Mechanisms for Bilateral Trading. *Journal of Economic Theory*,vol 29, 265-281, 1983.
- [16] Roundy R., Chen R., Janakriraman G. and Zhang R. Q. Efficient Auction Mechanisms for Supply Chain Procurement. *School of Operations Research and Industrial Engineering, Cornell University*,2001.
- [17] W. Vickrey Counterspeculation, Auctions and Competitive Sealed Tenders. *In Journal of Finance 1961*, vol. 16, pp. 8-37.

APPENDIX

A. CALCULATING THE OPTIMAL DIVIDING FUNCTION IN PROCUREMENT CLASS DOMAINS IN POLYNOMIAL TIME

In this section we show how to calculate the optimal dividing function for procurement class domains in polynomial time. We first define a special dividing function D'_0 which is easy to calculate:

We define the dividing function D'_0 recursively as follows: At stage j , D'_0 divides the trading players into two sets A_j and A'_j s.t.

- A_j is a procurement set
- A'_j can be divided into a disjoint union of procurement sets.
- A_j has minimal value from all possible such partitions.

Define $\mathbf{s}_j = A_j$ and recursively invoke D'_0 and A'_j until $A'_j = \emptyset$.

We now prove that D'_0 is the required dividing function.

LEMMA A.1. *For procurement class domains $D_0 = D'_0$.*

PROOF. Since the domain is a procurement class domain, for every reduced procurement set the set of players which achieve competition (either internal or external) is fixed.

Therefore, the number of procurement sets which are reduced is independent of the dividing function D . Since the goal is to optimize welfare by reducing procurement sets with the least value we can optimize welfare. This is achieved by D_0 . \square

B. PROBLEMS AND EXAMPLES

For completeness we present in this section the formal definitions of the problems that we use to illustrate our mechanism.

The first problem that we define is the *double-sided auction with homogeneous goods*.

PROBLEM B.1. Double-sided auction with homogeneous goods: *There are m sellers each of which have a single good (all goods are identical) and n buyers each of which are interested in receiving a good. We denote the set of sellers by S and the set of buyers by B . Every player $i \in S \cup B$ (both buyers and sellers) has a value v_i for the good. In this model a procurement set consists of a single buyer and a single seller, i.e., $|\mathbf{s}| = 2$. The value of a procurement set is $W(\mathbf{s}) = v_j - v_i$ where $j \in B$ and $i \in S$, i.e., the gain from trade.*

If procurement sets are created by matching the highest value buyer to the lowest value seller then [13]’s deterministic trade reduction mechanism¹⁷ reduces the lowest value procurement set.

A related model is the *pair related costs* [9] model.

PROBLEM B.2. The pair related costs: *A double-sided auction B.1 in which every pair of players $i \in S$ and $j \in B$ has a related cost $F(i, j) \geq 0$ in order to trade. $F(i, j)$ is a friction cost which should be minimized in order to maximize welfare.*

[9] defines two budget-balanced mechanisms for this case. One of [9]’s mechanisms has the set of buyers B as the X set for the X -external mechanism and the other has the set of sellers S as the X set for the X -external mechanism.

A similar model is the *spatially distributed markets (SDM)* model [3] in which there is a graph imposing relationships on the cost.

PROBLEM B.3. Spatially distributed markets: *there is a graph $G = (V, E)$ such that each $v \in V$ has a set of sellers S^v and a set of buyers B^v . Each edge $e \in E$ has an associated cost which is the cost to transport a single unit of good along the edge. The edges are non strategic but all players are strategic.*

[3] defines a budget balanced mechanism for this case. Our paper improves on [3] result.

Another graph model is the model defined in [6].

PROBLEM B.4. Trading Networks: *Given a graph and buyers and sellers who are situated on nodes of the graph. All trade must pass through a trader. In this case procurement sets are of the form (buyer, seller, trader) where the possible sets of this form are defined by a graph.*

The supply chain model [2, 4] can be seen as a generalization of [6] in which procurement sets consist of the form (producer, consumer, trader₁, ..., trader_k).

¹⁷It is also possible to randomize the reduction of procurements sets so as to achieve an expected budget of zero similar to [13], details are obvious and omitted.

PROBLEM B.5. Supply Chain: *There is a set D of agents and a set G of goods and a graph $G = (V, E)$ which defines possible trading relationships. Agents can require an input of multiple quantities of goods in order to output a single good. The producer type of player can produce goods out of nothing, the consumer has a valuation and an entire chain of interim traders is necessary to create a viable procurement set.*

[2, 4] consider *unique manufacturing technology* in which the graph defining possible relationships is a tree.

All of the above problems are procurement-class domains. We also consider several problems which are not procurement class domains and generally the questions of budget balance have been left as open problems.

An open problem raised in [3] is the SDM model in which edges are strategic.

PROBLEM B.6. Spatially distributed markets with strategic edges: *there is a graph $G = (V, E)$ such that each $v \in V$ has a set of sellers S^v and a set of buyers B^v . Each edge $e \in E$ has an associated cost which is the cost to transport a single unit of good along the edge. Each buyer, seller and edge has a value for the trade, i.e., all entities are strategic.*

[2, 4] left open the question of budget balanced mechanisms for supply chains where there is no unique manufacturing technology. It is easy to see that this problem is not a procurement class domain.

Another interesting problem is *transport networks*.

PROBLEM B.7. Transport networks: *A graph $G = (V, E)$ where the edges are strategic players with costs and the goal is to find a minimum cost transportation route between a pair of privileged nodes $Source, Target \in V$.*

It was shown in [1] that the efficient allocation can have a budget deficit that is linear in the number of players. Clearly, this problem is not a procurement class domain and [1] left the question of a budget balanced mechanism open.

Another non procurement-class based domain mechanism is the *double-sided combinatorial auction (CA) with single-value players*.

PROBLEM B.8. Double-sided combinatorial auction (CA) with single value players: *There exists a set S of sellers each selling a single good. There also exists a set B of buyers each interested in bundles of 2^S ¹⁸.*

There are two variants of this problem. In the single minded case each buyer has a positive value for only a single subset whereas in the multi minded case each buyer can have multiple bundles with positive valuation but all of the values are the same. In both cases we assume free disposal so that all bundles containing the desired bundle have the same value for the buyer.

We also consider problems that are non class domains.

PROBLEM B.9. Double-sided combinatorial auction (CA) with general multi-minded players: *same as B.8 but each buyer can have multiple bundles with positive valuation which are not necessarily the same.*

¹⁸We abuse notation and identify the seller with the good.

C. COMPARING DIFFERENT CHOICES OF X

The choice of X can have a large impact on the welfare (and revenue) of the reduced mechanism and therefore the question arises of how one should choose the set X .

As the X -external mechanism is required to maintain IC clearly the choice of X can not depend on the value of the players as otherwise the reduced mechanism will not be truthful.

In this section we motivate the choice of small X sets for procurement class domains and give intuition that it may also be the case for some other domains.

We start by illustrating the effect of the set X over the welfare and revenue in the double-sided auction with homogeneous goods problem B.1. Similar examples can be constructed for the other problems defined in B.

The following example shows an effect on the welfare.

EXAMPLE C.1. *There are two buyers and two sellers and two non intersecting (incomparable) sets $X = \{\text{buyers}\}$ and $Y = \{\text{sellers}\}$. If the values of the buyers are 101, 100 and the sellers are 150, 1 then the X -external mechanism will yield a gain from trade of 0 and the Y -external mechanism will yield a gain from trade of 100.*

Conversely, if the buyers values are 100, 1 and the sellers are 2, 3 the X -external mechanism will yield a gain from trade of 98 and the Y -external mechanism will yield a gain from trade of zero.

The example clearly shows that the difference between the X -external and the Y -external mechanism is unbounded although as shown above the fraction each of them reduces can be bound and therefore the multiplicative ratio between them can be bound (as a function of the number of trades).

On the revenue side we can not even bound the ratio as seen from the following example:

EXAMPLE C.2. *Consider k buyers with value 100 and $k + 1$ sellers with value 1.*

If $X = \{\text{buyers}\}$ then there is no need to reduce any trade and all of the buyer receive the good and pay 1. $k + 1$ of the sellers sell and each of them receive 1. This yields a net revenue of zero.

If $Y = \{\text{sellers}\}$ then one must reduce a trade! This means that all of the buyers pay 100 while all of the sellers still receive 1. the revenue is then $99k$.

Similarly, an example can be constructed that yields much higher revenue for the X -external mechanism as compared to the Y -external mechanism.

The above examples refer to sets X and Y which do not intersect and are incomparable. The following theorem compares the X -external and Y -external mechanisms for procurement class domains where X is a subset of Y .

THEOREM C.1. *For procurement class domains, if $X \subset Y$ and for any $s \in \mathbf{S}$, $s \cap X \cap Y \neq \emptyset$ then:*

1. *The efficiency of the X external mechanism in GTR-1 (and hence GTR-2) is at least that of the Y -external mechanism.*
2. *Any winning player that wins in both the X -external and Y -external mechanisms pays no less in the Y -external than in the X -external and therefore the ratio of budget to welfare is no worse in the Y external than the X -external.*

PROOF. 1. For any dividing function D if there is a procurement set s_j that is reduced in the X -external mechanism there are two possible reasons:

- (a) s_j lacks external competition in the X -external mechanism. In this case s_j lacks external competition in the internal mechanism.
- (b) s_j has all required external competitions in X -external. In this case s_j has all required internal competitions in Y -external by lemma 3.1 but might lack some external competition for $s_j \cup \{Y \setminus X\}$ and be reduced,

2. This follows from the fact that for any ordering D any procurement set s that is reduced in the X -external mechanism is also reduced in the Y -external mechanism. Therefore, the critical value is no less in the Y -external mechanism than the X -external mechanism.

□

REMARK C.1. *For any two sets X, Y it is easy to build an example in which the X -external and Y -external mechanisms reduce the same procurement sets so the inequality is weak.*

Theorem C.1 shows an inequality in welfare as well as for payments but it is easy to construct an example in which the revenue can increase for X as compared to Y as well as the opposite. This suggests that in general we want X to be as small as possible although in some domains it is not possible to compare different X 's.